

Hodge theory on hyperbolic manifolds of infinite volume

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Abstract

Let $X = H^n$ be the real hyperbolic space of dimension n , $G = SO(1, n)_0$ the identity component of its group of isometries, and let $\Gamma \subset G$ be a discrete torsion-free subgroup acting convex-cocompactly on X . We are going to study the (de Rham) cohomology groups $H^p(Y, \mathbb{C}) \cong H^p(\Gamma, \mathbb{C})$ of the hyperbolic manifold $Y = \Gamma \backslash X$. In this paper we describe certain spaces of generalized Γ -invariant currents on the sphere at infinity of X with support on the limit set of Γ . These spaces are finite-dimensional. The main result of the present paper identifies the cohomology with a quotient of such spaces (such a result was conjectured by Patterson [7]). We explain in which sense this result generalizes the classical Hodge Theorem for the case of compact quotients. We obtain analogous results for the groups $H^p(\Gamma, F)$, where F is an irreducible finite-dimensional representation of G .

1 Cocompact and convex-cocompact subgroups

Let $X = H^n$ be the real hyperbolic space of dimension n , and let ∂X be its geodesic boundary. The union $X \cup \partial X$ can be given the structure of an analytic manifold with boundary (use, e.g., the unit ball model of H^n) with an analytic action of the group $G = SO(1, n)_0$.

We consider a discrete torsion-free subgroup $\Gamma \subset G$. Its limit set $\Lambda \subset \partial X$ is defined to be the set of accumulation points in $X \cup \partial X$ of the Γ -orbit of an arbitrary point $x \in X$. The complement $\Omega := \partial X \setminus \Lambda$ is called the domain of discontinuity of Γ . Indeed, Γ acts properly discontinuously on Ω as well as on $X \cup \Omega$. Hence, $\overline{Y} := \Gamma \backslash (X \cup \Omega)$ is manifold with boundary $B := \Gamma \backslash \Omega$. Its interior $Y := \Gamma \backslash X$ carries the structure of a complete Riemannian manifold with constant sectional curvature -1 , i.e., a hyperbolic manifold. One calls Γ convex-cocompact if \overline{Y} is compact. In particular, a cocompact subgroup is convex-cocompact. In this case $\Lambda = \partial X$, $B = \emptyset$, and $\overline{Y} = Y$ is a closed hyperbolic manifold.

Our first goal is to reformulate the classical Hodge theorem for closed Riemannian manifolds in the special case of closed hyperbolic manifolds, i.e., cocompact subgroups $\Gamma \subset G$, in such a way that it still would make good sense for convex-cocompact non-cocompact Γ . The main tool will be the Poisson transform which sends differential forms on ∂X to harmonic forms on the interior. Let $\Omega^p(X)$, $\Omega_{-\omega}^p(\partial X)$ denote the spaces of smooth p -forms on X and p -forms with hyperfunction coefficients on ∂X , respectively. We consider the differential d , the codifferential $\delta = d^*$ defined by the hyperbolic metric, and the form Laplacian $\Delta = \delta d + d\delta$. Gaillard proved the following

Theorem 1 ([4]) *There are Poisson transforms which provide G -equivariant isomorphisms*

$$\Omega_{-\omega}^p(\partial X) \cong \{\omega \in \Omega^p(X) \mid \Delta\omega = 0, \delta\omega = 0\}, \quad p \neq \frac{n-1}{2},$$

and

$$\Omega_{-\omega}^{\pm}(\partial X) \cong \{\omega \in \Omega^{\frac{n-1}{2}}(X) \mid d\omega = 0, \delta\omega = 0\},$$

where $\Omega_{-\omega}^{\pm}(\partial X)$ are the eigenspaces of the conformally invariant $*$ -operator

$$* : \Omega^{\frac{n-1}{2}}(\partial X) \rightarrow \Omega^{\frac{n-1}{2}}(\partial X).$$

The theorem remains true if one replaces the left hand sides by the smaller spaces $\Omega_{-\infty}^*(\partial X)$ of forms with distribution coefficients, i.e., currents, and adds on the right hand sides the condition that the harmonic forms should have moderate growth (see [4]). Nowadays one version of the theorem can be obtained from the other by a "change of globalization functor" (see e.g. [3]).

By a slight abuse of notation we denote by $\Omega_{-\infty}^*(\Lambda) \subset \Omega_{-\infty}^*(\partial X)$ the space of all currents supported on the limit set of Γ . Γ acts on these spaces, and we denote by ${}^{\Gamma}\Omega_{-\infty}^*(\Lambda)$ and ${}^{\Gamma}\Omega_{-\infty}^*(\partial X)$ the corresponding spaces of Γ -invariants. Here is the promised reformulation of the Hodge Theorem.

Corollary 2 *If $Y = \Gamma \backslash X$ is a closed hyperbolic manifold, then there are the following isomorphisms*

$$\begin{aligned} H^p(Y, \mathbb{C}) &\cong {}^{\Gamma}\Omega_{-\infty}^{n-p}(\Lambda), \quad p \notin \{0, \frac{n+1}{2}\}, \\ H^{\frac{n+1}{2}}(Y, \mathbb{C}) &\cong {}^{\Gamma}\Omega_{-\infty}^{\pm}(\Lambda). \end{aligned}$$

Proof. Recall that in the cocompact case $\Lambda = \partial X$. Thus by Theorem 1 and the compactness of Y (in particular, moderate growth is automatic)

$${}^\Gamma \Omega_{-\infty}^*(\Lambda) \cong \{\omega \in \Omega^*(Y) \mid d\omega = 0, \delta\omega = 0\} \cong \{\omega \in \Omega^{n-*}(Y) \mid d\omega = 0, \delta\omega = 0\} .$$

The second isomorphism is given by the $*$ -operator. The right hand side is isomorphic to $H^{n-*}(Y, \mathbb{C})$ by the Hodge Theorem. \square

The unmotivated incorporation of the $*$ -operator in the isomorphism is necessary in order to produce a statement which has some chance to generalize to convex-cocompact Γ . Indeed, it turns out that Corollary 2 remains true in the general case for $p \geq \frac{n+1}{2}$. For $p \leq \frac{n}{2}$ sometimes a modification is necessary, see Theorem 7 below.

2 The extension map

From now on let $\Gamma \subset G$ be convex-cocompact. In this section we assume in addition that $\Omega \neq \emptyset$, i.e., Γ is not cocompact.

The aim of this section is to review the theory of the extension operator *ext* which extends Γ -invariant hyperfunction sections defined on Ω of homogeneous vector bundles over ∂X across the limit set in order to get globally defined ones. This theory was developed by U. Bunke and the author in [1] and [2]. It will enable us to get hold on the spaces ${}^\Gamma \Omega_{-\infty}^p(\Lambda)$ and its relatives defined below.

As a homogeneous space we have $\partial X = G/P$, where $P = MAN$ is the group of orientation preserving conformal transformations of \mathbb{R}^{n-1} leaving the point ∞ fixed. Here $M = SO(n-1)$, $A \cong \mathbb{R}_+$ is the group of dilatations, and N is the group of translations. Let \mathfrak{a} be the Lie algebra of A . We will identify its complexified dual $\mathfrak{a}_{\mathbb{C}}^*$ with \mathbb{C} . Let \hat{M} be the set of equivalence classes of irreducible representations of M in complex vector spaces. Any pair $(\sigma, \lambda) \in \hat{M} \times \mathfrak{a}_{\mathbb{C}}^*$ gives rise to an irreducible representation σ_λ of P in the representation space of σ by

$$\sigma_\lambda(man) := \sigma(m)a^{\rho-\lambda} , \quad \rho = \frac{n-1}{2} .$$

Let $V(\sigma_\lambda) \rightarrow \partial X$ be the associated homogeneous vector bundle. By $C^{-\omega}(\partial X, V(\sigma_\lambda))$ we denote the space of its hyperfunction sections. It carries a representation of G known under the name principal series representation. Let σ^p be the p -th exterior power of the standard representation of $SO(n-1)$. Then we have

$$\Omega_{-\omega}^p(\partial X) = C^{-\omega}(\partial X, V(\sigma_{\rho-p}^p)) .$$

We want to understand the space of Γ -invariant sections ${}^\Gamma C^{-\omega}(\partial X, V(\sigma_\lambda))$, in particular its subspace ${}^\Gamma C^{-\omega}(\Lambda, V(\sigma_\lambda))$ of sections supported on the limit set. Since $\Gamma \backslash \Omega$ is a compact manifold the space of invariant sections along Ω is easy to understand:

$${}^\Gamma C^{-\omega}(\Omega, V(\sigma_\lambda)) \cong C^{-\omega}(B, V_B(\sigma_\lambda)) .$$

It is the space of hyperfunction sections of the corresponding bundle $V_B(\sigma(\lambda))$ on B . Restriction of a hyperfunction on ∂X to the open set Ω provides a map

$$res : {}^\Gamma C^{-\omega}(\partial X, V(\sigma_\lambda)) \rightarrow C^{-\omega}(B, V_B(\sigma_\lambda)) .$$

It turns out that for generic $\lambda \in \mathfrak{a}_\mathbb{C}^*$ the map res is an isomorphism. Moreover, res has an inverse which depends meromorphically on λ :

Theorem 3 ([2], [1]) *There is a meromorphic family of maps with finite-dimensional singularities*

$$ext_\lambda : C^{-\omega}(B, V_B(\sigma_\lambda)) \rightarrow {}^\Gamma C^{-\omega}(\partial X, V(\sigma_\lambda))$$

such that

$$res \circ ext_\lambda = \text{id} . \quad (1)$$

A singularity at λ is called finite-dimensional if the coefficients in the principal part of the Laurent expansion at λ are finite-dimensional operators. The theorem is proved in [2] in the distribution setting, only. But the arguments of [1], 2.5. and 2.6. show how one can transfer it to hyperfunctions.

In order to deal with points $\lambda \in \mathfrak{a}_\mathbb{C}^*$, where ext has a pole, we have to introduce slightly more general bundles. We consider the representation $1^k : MAN \rightarrow \mathbb{C}^k$ given by the Jordan block

$$1^k(man) = \exp \begin{pmatrix} 0 & \log(a) & 0 & \dots & 0 \\ 0 & 0 & \log(a) & \dots & 0 \\ & & \ddots & & \\ 0 & 0 & \dots & 0 & \log(a) \\ 0 & 0 & \dots & 0 & 0 \end{pmatrix} .$$

Then we set $V^k(\sigma_\lambda) := V(\sigma_\lambda \otimes 1^k)$. Obviously, $V^1(\sigma_\lambda) = V(\sigma_\lambda)$, and for any $l \in \mathbb{N}$ there is a short exact sequence

$$0 \rightarrow V^k(\sigma_\lambda) \xrightarrow{i_l} V^{k+l}(\sigma_\lambda) \xrightarrow{p_k} V^l(\sigma_\lambda) \rightarrow 0 .$$

Again we have a restriction map between the corresponding spaces of sections

$$res : {}^\Gamma C^{-\omega}(\partial X, V^k(\sigma_\lambda)) \rightarrow C^{-\omega}(B, V_B^k(\sigma_\lambda)) .$$

The space of sections of the extended bundle $C^{-\omega}(\partial X, V^k(\sigma_\lambda))$ can be G -equivariantly identified with the quotient of the space of germs at λ of meromorphic families $\mu \mapsto f_\mu \in C^{-\omega}(\partial X, V(\sigma_\mu))$ which have a pole of order at most k by the space of germs of holomorphic families. By choosing a holomorphic identification of the bundles $V_B(\sigma_\mu)$ for varying μ with $V_B(\sigma_\lambda)$ we can think of $V_B^k(\sigma_\lambda)$ as a subspace of the germs at λ

of meromorphic families $\mu \mapsto f_\mu \in C^{-\omega}(B, V_B(\sigma_\mu))$. These interpretations show that for any $k \in \mathbb{N}$ the family of operators ext_μ induces a map

$$ext^k : C^{-\omega}(B, V_B^k(\sigma_\lambda)) \rightarrow {}^\Gamma C^{-\omega}(\partial X, V^{k+k_-}(\sigma_\lambda)) ,$$

where $k_- = k_-(\sigma_\lambda)$ denotes the order of the pole of ext at λ . By (1) it satisfies $res \circ ext^k = i_{k_-}$. Since res commutes with p_k we obtain

$$res \circ p_k \circ ext^k = 0 . \quad (2)$$

We are interested in the spaces ${}^\Gamma C^{-\infty}(\Lambda, V^k(\sigma_\lambda))$ of Γ -invariant distribution sections of $V^k(\sigma_\lambda)$ supported on Λ which generalize the spaces of currents ${}^\Gamma \Omega_{-\infty}^p(\Lambda)$ discussed before.

Showing that Poisson transforms of elements of ${}^\Gamma C^{-\omega}(\Lambda, V^k(\sigma_\lambda))$ are always of moderate growth on X one obtains

Lemma 4 ${}^\Gamma C^{-\infty}(\Lambda, V^k(\sigma_\lambda)) = {}^\Gamma C^{-\omega}(\Lambda, V^k(\sigma_\lambda))$.

Combining the lemma with Equation (2) we see that $p_k \circ ext^k$ maps $C^{-\omega}(B, V_B^k(\sigma_\lambda))$ to ${}^\Gamma C^{-\infty}(\Lambda, V^{k_-}(\sigma_\lambda))$. We denote the image of $p_{k_-} \circ ext^{k_-}$ by $E_\Lambda^+(\sigma_\lambda)$. In fact, it consists of all invariant distributions on the limit set which can be constructed by means of the extension map.

Along the lines of the proofs of Proposition 6.11 and Corollary 6.12 in [2] one shows in addition

Proposition 5 *The sequence of inclusions*

$${}^\Gamma C^{-\infty}(\Lambda, V^1(\sigma_\lambda)) \subset \dots \subset {}^\Gamma C^{-\infty}(\Lambda, V^k(\sigma_\lambda)) \subset {}^\Gamma C^{-\infty}(\Lambda, V^{k+1}(\sigma_\lambda)) \subset \dots$$

stabilizes at some $k =: k_+(\sigma_\lambda)$, and $\dim {}^\Gamma C^{-\infty}(\Lambda, V^{k_+}(\sigma_\lambda)) < \infty$.

3 The main theorem

We now want to relate the spaces ${}^\Gamma C^{-\infty}(\Lambda, V^k(\sigma_{p-1-\rho}^{n-p}))$ to cohomology. This can be done by the following proposition which is again a corollary of Gaillard's results [4].

Proposition 6 *Suitably defined Poisson transforms P_p^k, P_\pm^k provide G -equivariant isomorphisms*

$$P_p^k : C^{-\omega}(\partial X, V^k(\sigma_{p-1-\rho}^{n-p})) \xrightarrow{\cong} \{\omega \in \Omega^p(X) \mid \Delta^k \omega = 0, d\omega = 0\} , \quad p \neq \frac{n+1}{2} ,$$

and

$$P_\pm^k : C^{-\omega}(\partial X, V^k(\sigma_0^\pm)) \xrightarrow{\cong} \{\omega \in \Omega^{\frac{n+1}{2}}(X) \mid (d*)^k \omega = 0, d\omega = 0\} .$$

Note that the Poisson transforms P_p^1 differ from the ones in Theorem 1 by the application of the $*$ -operator. We are now able to state the main theorem of the paper which generalizes Corollary 2 to arbitrary convex-cocompact groups.

Theorem 7 *For $p \in \{1, 2, \dots, n\}$ set $l(p) := k_+(\sigma_{p-1-\rho}^{n-p})$. Then $l(p) \leq k_-(\sigma_{p-1-\rho}^{n-p}) + 1$, and for $k \geq l(p)$ the Poisson transforms P_p^k , P_\pm^k induce isomorphisms*

$$\begin{aligned} H^p(Y, \mathbb{C}) &\cong {}^\Gamma C^{-\infty}(\Lambda, V^k(\sigma_{p-1-\rho}^{n-p}))/E_\Lambda^+(\sigma_{p-1-\rho}^{n-p}), \quad p \neq \frac{n+1}{2}, \\ H^{\frac{n+1}{2}}(Y, \mathbb{C}) &\cong {}^\Gamma C^{-\infty}(\Lambda, V^k(\sigma_0^\pm))/E_\Lambda^+(\sigma_0^\pm). \end{aligned}$$

For $p \geq \frac{n+1}{2}$ we have $l(p) \leq 1$ and the following isomorphisms

$$H^p(Y, \mathbb{C}) \cong \begin{cases} {}^\Gamma \Omega_{-\infty}^{n-p}(\Lambda), & p > \frac{n+1}{2} \\ {}^\Gamma \Omega_{-\infty}^\pm(\Lambda), & p = \frac{n+1}{2} \end{cases} \quad (3)$$

$$\cong \{\omega \in {}^\Gamma \Omega_{-\infty}^{n-p}(\Lambda) \mid d\omega = 0\}. \quad (4)$$

Let us add a couple of remarks concerning the theorem.

- If Γ is cocompact then it is natural to set $E_\Lambda^+(\sigma_{p-1-\rho}^{n-p}) := \{0\}$ and $k_-(\sigma_{p-1-\rho}^{n-p}) = 0$. With these conventions Theorem 7 applied to cocompact Γ is just a reformulation of Corollary 2.
- For $p > \frac{n+1}{2}$ the theorem is a fairly direct consequence of the results of Mazzeo/Philipps [6] concerning the L^2 -cohomology of Y . This was already noted by Patterson [7] and was also observed by Lott [5]. But for $p < \frac{n+1}{2}$ L^2 -methods are not sufficient.
- That the cohomology of Y (as well as $H^p(\Gamma, F)$ considered in the next section) should be representable by currents on the limit set was conjectured by Patterson [7]. In fact, he suggested that the isomorphism (4) should be true for all p . But there are examples where the map $\{\omega \in {}^\Gamma \Omega_{-\infty}^{n-p}(\Lambda) \mid d\omega = 0\} \rightarrow H^p(Y, \mathbb{C})$ is not injective. We do not know whether this map is surjective in general.

We will now describe the main ideas entering the proof of Theorem 7. Full details and several related results and applications will appear in a forthcoming paper.

Using the surjectivity of the Laplacian on $\Omega^p(Y)$ one shows that any cohomology class can be represented by a closed form which is annihilated by some power of the Laplace operator (in fact even by a coclosed harmonic form). Then the idea was that such a form is exact if and only if it arises as a value or a derivative at $\lambda = 0$ of a family of closed eigenforms ω_λ , $\Delta\omega_\lambda = \lambda\omega_\lambda$. Using the theory of Poisson transforms one translates the problem to the boundary. Families of eigenforms correspond to families of Γ -invariant hyperfunction sections of the corresponding bundles over ∂X . Such families can always be described using ext_λ . Now one uses the extension operator

in order to produce a representative of each cohomology class which is supported on the limit set.

From the technical point of view it is simpler to replace this family criterion by the following infinitesimal version: a closed form is exact if and only if it belongs to the image of Δ^k restricted to closed forms for some or, equivalently, for all k . Applying Proposition 6 and observing that the action of the Laplacian on closed forms annihilated by Δ^k corresponds to the action of p_1 on ${}^\Gamma C^{-\omega}(\partial X, V^k(\sigma_{p-1-\rho}^{n-p}))$ we are lead to the setting of Section 2. Using Proposition 5 one shows that an element is in the image of p_1 if and only if it is in the image of some ext^l or $p_m \circ ext^l$. Eventually we end up with Theorem 7.

4 Finite-dimensional G -representations as coefficients

We can interpret the cohomology of Y as the group cohomology of Γ with coefficients in its trivial representation: $H^p(Y, \mathbb{C}) \cong H^p(\Gamma, \mathbb{C})$. If F is a finite-dimensional representation of G , then we can look at it as a representation of Γ and form $H^p(\Gamma, F)$. Since any finite-dimensional G -representation is semisimple we can assume that F is irreducible. It turns out that the theory explained in the previous two sections is well-behaved with respect to the translation functor, i.e., tensoring with F followed by the projection according to the generalized infinitesimal character of F . This allows one to translate Theorem 7 to $H^*(\Gamma, F)$ for any irreducible F . In order to formulate the corresponding theorem we need some preparation.

Denote by \mathfrak{g} and \mathfrak{m} the Lie algebras of G and M , respectively. Let $\mathfrak{t} \subset \mathfrak{m}$ be the Lie algebra of a maximal torus of M . Then $\mathfrak{h} := \mathfrak{a} \oplus \mathfrak{t}$ is a Cartan subalgebra of \mathfrak{g} . We fix a positive Weyl chamber $\mathfrak{t}_+^* \subset i\mathfrak{t}^*$ and denote by $\rho_{\mathfrak{m}} \in \mathfrak{t}_+^*$ the half sum of the corresponding positive roots of \mathfrak{m} . As a positive Weyl chamber $\mathfrak{h}_+^* \subset \mathfrak{h}_{\mathbb{R}}^* := \mathfrak{a}^* \oplus i\mathfrak{t}^*$ we choose the one which contains \mathfrak{a}_+^* and is contained in $\mathfrak{a}_+^* \oplus \mathfrak{t}_+^*$. We consider the following subset of the Weyl group $W(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})$ of $\mathfrak{g}_{\mathbb{C}}$:

$$W^1 := \{w \in W(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}}) \mid \alpha \in \mathfrak{h}_+^* \Rightarrow w\alpha|_{\mathfrak{t}} \in \mathfrak{t}_+^*\} .$$

If n is even then $W^1 = \{w_0, w_1, \dots, w_{n-1}\}$, where w_i is the unique element of length i in W^1 . For odd n we have $W^1 = \{w_0, \dots, w_{\frac{n-3}{2}}, w_+, w_-, w_{\frac{n+1}{2}}, \dots, w_{n-1}\}$, where w_i and w_{\pm} have length i and $\frac{n-1}{2}$, respectively.

For a finite-dimensional irreducible representation F with highest weight $\nu \in \mathfrak{h}_+^*$ we define $\sigma_F^p \in \hat{M}$ by its highest weight

$$\mu_p := w_{n-1-p}(\nu + \rho + \rho_{\mathfrak{m}})|_{\mathfrak{t}} - \rho_{\mathfrak{m}} \in \mathfrak{t}_+^* .$$

For odd n we can replace w_p by w_{\mp} and obtain the highest weights of representations $\sigma_F^{\pm} \in \hat{M}$. Set $\sigma_F^{\frac{n-1}{2}} := \sigma_F^+ \oplus \sigma_F^-$. Furthermore we define

$$\lambda_p := -w_{n-1-p}(\nu + \rho + \rho_{\mathfrak{m}})|_{\mathfrak{a}} \in \mathfrak{a}^* .$$

The analogously defined elements λ_{\pm} are equal to 0.

Note that for $F = \mathbb{C}$, i.e., $\nu = 0$, we have $\sigma_{\mathbb{C}, \lambda_p}^p = \sigma_{\rho-p}^p$. For general F the parameters σ_{F, λ_p}^p explicitly describe the effect of the translation functor applied to the de Rham complex of ∂X . Indeed, the spaces $C^{-\omega}(\partial X, V(\sigma_{F, \lambda_p}^p))$ fit into an exact de Rham like complex of G -equivariant differential operators

$$\begin{aligned} 0 \rightarrow F \rightarrow C^{-\omega}(\partial X, V(\sigma_{F, \lambda_0}^0)) &\xrightarrow{d_F^0} C^{-\omega}(\partial X, V(\sigma_{F, \lambda_1}^1)) \xrightarrow{d_F^1} \\ \dots &\xrightarrow{d_F^{n-2}} C^{-\omega}(\partial X, V(\sigma_{F, \lambda_{n-1}}^{n-1})) \rightarrow F \rightarrow 0 \end{aligned}$$

which appears in the literature under various names like BGG-resolution or Želobenko complex. Generalizations of it also played a central role in J. Slovák's talk at the Workshop. The following theorem is the natural generalization of Theorem 7.

Theorem 8 *For a finite-dimensional irreducible representation F of G and $p \in \{1, 2, \dots, n\}$ set $l(F, p) := k_+(\sigma_{F, \lambda_{n-p}}^{n-p})$. Then $l(p) \leq k_-(\sigma_{F, \lambda_{n-p}}^{n-p}) + 1$, and for $k \geq l(F, p)$ there are isomorphisms*

$$\begin{aligned} H^p(\Gamma, F) &\cong {}^\Gamma C^{-\infty}(\Lambda, V^k(\sigma_{F, \lambda_{n-p}}^{n-p}))/E_\Lambda^+(\sigma_{F, \lambda_{n-p}}^{n-p}), \quad p \neq \frac{n+1}{2}, \\ H^{\frac{n+1}{2}}(\Gamma, F) &\cong {}^\Gamma C^{-\infty}(\Lambda, V^k(\sigma_{F, 0}^\pm))/E_\Lambda^+(\sigma_{F, 0}^\pm). \end{aligned}$$

For $p \geq \frac{n+1}{2}$ we have $l(p) \leq 1$ and the following isomorphisms

$$\begin{aligned} H^p(\Gamma, F) &\cong \begin{cases} {}^\Gamma C^{-\infty}(\Lambda, V(\sigma_{F, \lambda_{n-p}}^{n-p})) & p > \frac{n+1}{2} \\ {}^\Gamma C^{-\infty}(\Lambda, V(\sigma_{F, 0}^\pm)) & p = \frac{n+1}{2} \end{cases} \\ &\cong \left\{ \omega \in {}^\Gamma C^{-\infty}(\Lambda, V(\sigma_{F, \lambda_{n-p}}^{n-p})) \mid d_F^{n-p} \omega = 0 \right\}. \end{aligned}$$

All these isomorphisms are induced by Poisson transforms into closed forms with values in the flat bundle over Y corresponding to F .

Acknowledgements

Discussions with the following mathematicians have strongly influenced my changing view on the subject over the years: A. Juhl, S. J. Patterson, U. Bunke and P.-Y. Gaillard. Let me thank all of them. In addition, I want to thank Joachim Hilgert who invited me to present the above results at the Workshop "Lie Theory and Its Applications in Physics - Lie III".

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